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Uniform Approximation of Nonnegative Continuous Linear Functionals[1]

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Rogosinski has developed a theory for nonnegative linear functionals in finite dimensional real spaces. Later he extended this theory to the infinite dimensional case via weak topologies. In this paper, we extend the theory to complex valued functions on compact sets, and to a more restrictive class of infinite dimensional spaces, utilizing only norm topologies. Although more restrictive than that of Rogosinski, this theory is more amenable to numerical analysis applications.

The key theorem, Theorem 2.2, gives some sufficient conditions for uniform approximation of a nonnegative linear functional by finite *nonnegative* combinations of function evaluations. The third section gives some finite dimensional applications.

1. INTRODUCTION

Let T be an arbitrary set, and $C_\infty(T)$ a linear space of (real or complex valued) functions defined on T. Let the convex cone P be defined by

$$P \equiv \{f \in C_\infty(T) \,|\, \mathrm{Re}\{f(t)\} \geqslant 0,\ t \in T\}.$$

A linear functional L on $C_\infty(T)$ is said to be *nonnegative* if $\mathrm{Re}\{L(f)\} \geqslant 0$, $\forall f \in P$.

For any $t \in T$, the *point functional* L_t *at t, defined* by $L_t(f) = f(t)$, is clearly a nonnegative linear functional, so that the hull cone of the set $F \equiv \{L_t \,|\, t \in T\}$ of point functionals, is clearly contained in the cone of nonnegative functionals. The precise relationship between these two cones is of interest in numerical analysis, since it tells us when a particular nonnegative linear functional can be approximated by nonnegative linear combinations of function evaluations.

In [*6*], Rogosinski summarizes results known for real, finite dimensional $C_\infty(T)$, giving conditions for these cones to be identical. In [*7*], he discusses the infinite dimensional real case, using weak topologies on the algebraic dual of $C_\infty(T)$. Here, we shall restrict ourselves to reflexive Banach spaces, utilizing only norm topologies and continuous linear functionals. We state, however,

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results for both the real and the complex cases. Further, we do not require the concept of moment cone, since we are not concerned explicitly with moment problems, another area of application for this type of theory.

Our approach leads to the following theorems, useful in numerical analysis.

THEOREM 2.2. *Let $C_\infty(T)$ be a reflexive Banach space, in which point functionals are continuous. Let M be a nonnegative linear functional. Then, $\forall\ \epsilon > 0$, there exist points $t_1, t_2,\ldots, t_N$ in T, and positive scalars $\lambda_1, \lambda_2,\ldots, \lambda_N$, such that*

$$\left|M(f) - \sum_{i=1}^{N} \lambda_i f(t_i)\right| < \epsilon\|f\|, \qquad \forall\, f \in C_\infty(T).$$

THEOREM 3.1. *Let $C_n(T)$ be the span of n continuous, linearly independent functions, defined on a compact set T. We assume that $\exists\, f \in C_n(T)$ such that $f(t) > 0$ on T. If M is a nonnegative linear functional, then there exist points $t_1, t_2,\ldots, t_N$, and positive scalars $\lambda_1, \lambda_2,\ldots, \lambda_N$, such that*

$$M(f) = \sum_{i=1}^{N} \lambda_i f(t_i), \qquad \forall\, f \in C_n(T),$$

where $N \leqslant n$ or $N \leqslant 2n$, depending on whether the functions are real or complex valued.

This last theorem is known in the real case, and appears explicitly in Rogosinski [*6*] and some of his earlier papers, and implicitly in Tchakaloff [*8*], who applied it to quadratures, and pointed out its usefulness in numerical analysis. In Section 2, we characterize the two cones mentioned, prove Theorem 2.2, and show an application of the theorem. In Section 3, we prove Theorem 3.1, and indicate some finite dimensional applications.

We require a few definitions, and some notation. Let X be a Banach space. By $K(S)$, S a given set, we mean the *hull cone* of S, the smallest convex cone (with 0 as vertex) containing S. Analogous with convex hulls, we know that $K(S)$ is the set of all finite nonnegative combinations of elements of S. (See Wilansky [*9*], page 32.)

For a given cone K, we define the *dual* or *polar* cone as

$$K^\oplus \equiv \{L \in X^* \mid \operatorname{Re}\{L(x)\} \geqslant 0,\ \forall\ x \in K\},$$

where X^* is the conjugate or normed dual of X. Clearly, $K^\oplus$ is a closed convex cone. Further, if we define the *dual* or *polar cone* of a *set* S,

$$S^\oplus \equiv \{L \in X^* \mid \operatorname{Re}\{L(x)\} \geqslant 0,\ \forall\ x \in S\},$$

then $S^\oplus$ is also a closed convex cone. It is easily shown that $S^\oplus = K(S)^\oplus$.

Finally, we require the following separation theorem given in Wilansky [9].

THEOREM 1.1. *Let X be a locally convex linear space, A a convex set, $x_0 \notin \bar{A}$ a point in X. Then x_0 and A can be strictly separated. That is, $\exists\, x' \in X^*$, such that*

$$\operatorname{Re} x'(x_0) > \sup\{\operatorname{Re} x'(x) \,|\, x \in A\}.$$

2. NONNEGATIVE FUNCTIONALS AND REFLEXIVE SPACES

Recall that for a given set T, $C_\infty(T)$ is a Banach space of real or complex valued functions defined on T, and P is the cone of nonnegative functions. Clearly, $P^\oplus$, the dual of P, is the cone of continuous nonnegative linear functionals. We shall assume that the set F of all point functionals is contained in $[C_\infty(T)]^*$. That is, point functionals are continuous. Then, immediately,

$$K(F) \subset P^\oplus,$$

and, in fact, $\overline{K(F)} \subseteq P^\oplus$, since $P^\oplus$ is a closed convex cone.

THEOREM 2.1. *If $C_\infty(T)$ is reflexive, then*

$$\overline{K(F)} = P^\oplus.$$

Proof. We need only show $P^\oplus \subseteq \overline{K(F)}$; so assume $L_0 \in P^\oplus$, and $L_0 \notin \overline{K(F)}$. By Theorem 1.1, there exists $Z \in [C_\infty(T)]^{**}$ such that

$$\operatorname{Re} Z(L_0) < \inf\{\operatorname{Re} Z(L) \,|\, L \in \overline{K(F)}\} = \alpha.$$

Now, $0 \in \overline{K(F)}$, so $\alpha \leqslant 0$. Suppose $\alpha < 0$. Then, $\exists\, L_1 \in \overline{K(F)}$ such that $\operatorname{Re} Z(L_1) = \alpha_1$, where $0 > \alpha_1 > \alpha$. Since $L_1 \in \overline{K(F)}$, a convex cone, $nL_1 \in \overline{K(F)}$, $\forall\, n = 1, 2, \ldots$. Thus,

$$\alpha \leqslant \operatorname{Re} Z(nL_1) = n \operatorname{Re} Z(L_1) = n\alpha_1, \qquad n = 1, 2, \ldots,$$

a contradiction for n large enough (since α and α_1 are negative).

Thus,

$$\operatorname{Re} Z(L_0) < 0,$$

and

$$\operatorname{Re} Z(L) \geqslant 0 \;\forall\, L \in \overline{K(F)}.$$

Now, let $f \in C_\infty(T)$ correspond to Z in the second dual. Then,

$$\operatorname{Re}\{f(t)\} = \operatorname{Re}\{Z(L_t)\} \geqslant 0,$$

so that $f(t) \in P$. By assumption, $L_0 \in P^\oplus$, so $\operatorname{Re}\{L_0(f)\} = \operatorname{Re}\{Z(L_0)\} \geqslant 0$, a contradiction. Hence, $L_0 \in \overline{K(F)}$. Q.E.D.

This allows us to prove the following theorem on uniform approximation.

THEOREM 2.2. *Let $C_\infty(T)$ be a reflexive Banach space, in which point functionals are continuous. Let M be a nonnegative linear functional. Then, $\forall\ \epsilon > 0$, there exist points $t_1, t_2, \ldots, t_N$ in T, and positive scalars $\lambda_1, \lambda_2, \ldots, \lambda_N$, such that*

$$\left| M(f) - \sum_{i=1}^{N} \lambda_i f(t_i) \right| < \epsilon \| f \|, \qquad \forall f \in C_\infty(T).$$

Proof. $M \in P^\oplus = \overline{K(F)}$, so, for a given positive ϵ, there is a positive combination of point functionals,

$$\sum_{i=1}^{N} \lambda_i L_{t_i}, \qquad \lambda_i \in (0, \infty), \qquad t_i \in T,$$

such that

$$\left\| M - \sum_{i=1}^{N} \lambda_i L_{t_i} \right\| < \epsilon.$$

Then

$$\left| M(f) - \sum_{i=1}^{N} \lambda_i f(t_i) \right| = \left| \left(M - \sum_{i=1}^{N} \lambda_i L_{t_i} \right)(f) \right| < \epsilon \| f \|, \qquad \forall f \in C_\infty(T).$$

Q.E.D.

Note that we have a uniform approximation by a linear combination of point functionals, with positive coefficients, for both the real and the complex case.

In the preceding two theorems, the only restrictions we have imposed were on the norm. We have required a reflexive space, where the point functionals are continuous. We have made no restriction on T, or on the functions.

We give an example now of an infinite dimensional space to which Theorem 2.2 may be applied. For $C_\infty(T)$, take the complex Hilbert space $L_2(B)$ given in Davis [*1*], where B is an open connected set in the complex plane. The point functionals are continuous in this space. Suppose S is a rectifiable arc within B. The functional

$$M(f) = \int_S f(z)\, ds$$

is a nonnegative linear functional, and hence, can be uniformly approximated by nonnegative combinations of function values.

3. Finite Dimensional Applications

For cones in finite dimensional spaces, we can show some additional results. It is well known that if S is any set in n-dimensional real space, $K(S)$ consists of all nonnegative combinations of at most n elements of F. See Tchakaloff [*8*] or Fenchel [*3*].

The map $T: C^n \to E^{2n}$, taking n-dimensional complex space into $2n$-dimensional real space, given by

$$T((x_1 + iy_1, \ldots, x_n + iy_n)) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$$

is 1–1, onto, and bicontinuous. Further, for α, β real,

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v).$$

Hence, we have immediately,

LEMMA 3.1. *Let C be a non-empty set in C^n. Then*

(i) *C is convex* iff *$T(C)$ is convex,*
(ii) $T(H(C)) = H(T(C))$,
(iii) $T(K(C)) = K(T(C))$,

where $H(C)$ denotes the convex hull of a set C.

Thus, for a complex n-dimensional space, any element in $K(S)$ can be expressed as a nonnegative combination of at most $2n$ elements of S. Further, the well-known corollary of Caratheodory's theorem (stating that the convex hull of a compact set is compact), holds for n-dimensional complex space, as well as for n-dimensional real space.

Let $\Phi_1(t), \ldots, \Phi_n(t)$ be defined and linearly independent on T, and denote their span by $C_n(T)$. Let P_n denote the cone of nonnegative functions. Its finite dimensionality implies that the space is reflexive, and point functionals are continuous, so that, by Theorem 2.2, $\overline{K(F)} = P^{\oplus}$, where F is the set of point functionals. Further, if $M \in K(F)$, then M has the form

$$M = \sum_{i=1}^{N} \lambda_i L_{t_i}, \qquad \lambda_i > 0, \qquad t_i \in T,$$

where $N \leqslant n$ or $N \leqslant 2n$, depending on whether the functions are real or complex valued.

We are concerned with conditions which imply $K(F) = P^{\oplus}$. For this purpose, we define the projection cone $U(C)$ of a given set C, as the cone

$$U(C) \equiv \{\lambda C \mid \lambda \in [0, \infty)\}.$$

LEMMA 3.2.

(i) *C convex $\Rightarrow$ $U(C)$ convex.*
(ii) $C \subseteq U(C) \subseteq K(C)$.
(iii) *C convex $\Rightarrow$ $U(C) = K(C)$.*
(iv) $K(C) = K(H(C)) = U(H(C))$,

where $H(C)$ is the convex hull of C.

We omit the proof, as it is quite easy.

LEMMA 3.3. *If C is a closed bounded set, not containing* 0, *then $U(C)$ is closed.*

Krasnoselskii [5] shows this for real Banach spaces, but the proof carries over directly to the complex case. The last two lemmas hold for infinite dimensional spaces, but the application to finite dimensional spaces is particularly easy. If F is compact, $H(F)$ is compact, and if $0 \notin H(F)$, then $K(F)$ is closed, so that $K(F) = P_n^{\oplus}$.

We shall assume the following generalized Krein condition (Rogosinski [6]), namely, $\exists\, p(t) \in P_n$ such that $\mathrm{Re}\{p(t)\} \geqslant \alpha > 0$ on T. This is no loss in generality, since we can always append a constant function to the set, if the condition is not satisfied.

It is well known that if T is a compact set in E^n, and $\Phi_1, \ldots, \Phi_n$ are continuous real functions, then $K(F)$ is closed. We show this for the complex case as well.

LEMMA 3.4. *If $\Phi_1, \ldots, \Phi_n$ are continuous, and if the generalized Krein condition is satisfied, then, if T is compact, so is F, and $0 \notin H(F)$. Thus, $K(F)$ is a closed set.*

Proof. If we let $p(t) \in C_n(T)$ be represented by

$$p(t) = \sum_{i=1}^{n} A_i \Phi_i(t),$$

and if L is a linear functional,

$$L(p) = \sum_{i=1}^{n} A_i L\Phi_i,$$

then, the imbedding of $[C_n(T)]^*$ into E^n or C^n, as the case may be, associates with each L, the vector $(L\Phi_1, \ldots, L\Phi_n)$. The point functional L_t is associated with $(\Phi_1(t), \ldots, \Phi_n(t))$. This is a continuous map of T into E^n or C^n, so the set of point functionals is compact. Since there is no $t^* \in T$ such that $\Phi_i(t^*) = 0$, $\forall\, i$, therefore $0 \notin F$. For $L \in H(F)$, let

$$L = \sum_{i=1}^{N} \lambda_i L_{t_i}, \qquad \lambda_i > 0, \qquad \sum_{i=1}^{N} \lambda_i = 1.$$

Let $p(t)$ be a function such that $\mathrm{Re}\{p(t)\} \geqslant \alpha > 0$. Then

$$L(p) = \sum_{i=1}^{N} \lambda_i\, p(t_i),$$

and $\mathrm{Re}\{L(p)\} \geqslant \alpha > 0$. Now, $0 < |L(p)| \leqslant \|L\| \cdot \|p\|$, so $\|L\| > 0$, and $L \not\equiv 0$. Hence, $0 \notin H(F)$. By the remarks above, $K(F)$ is closed. Q.E.D.

THEOREM 3.1. *Let $C_n(T)$ be the span of n continuous, linearly independent functions, defined on a compact set T, and suppose $C_n(T)$ satisfies the generalized Krein condition. Let M be a nonnegative linear functional. Then, $K(F) = P^{\oplus}$,* *and there exist points* $t_1, \ldots, t_N$, *and positive scalars* $\lambda_1, \ldots, \lambda_N$, *such that*

$$M(f) = \sum_{i=1}^{N} \lambda_i \, f(t_i) \;\; \forall f \in C_n(T),$$

where $N \leqslant n$ *or* $N \leqslant 2n$, *depending on whether the functions are real or complex valued.*

The reverse implication here is obvious. If such an expansion exists, then M is a nonnegative linear functional.

Example 3.1. Let $\Phi_1, \ldots, \Phi_n$ be real, continuous, and linearly independent on $T \subseteq E^r$, T compact, and assume the Krein condition is satisfied. Then, if $w(t) > 0$ on T, there exists a quadrature formula

$$\int_T w(t) f(t) dt = \sum_{i=1}^{n} \lambda_i f(t_i),$$

where $\lambda_i \geqslant 0$, $t_i \in T$, exact for all $f \in C_n(T)$.

Example 3.2. Let T be a rectifiable arc in E^3, let $C_n(T)$ be the span of n real, continuous, linearly independent functions on T, and assume the Krein condition is satisfied. Then there exists a quadrature formula for the line integral,

$$\int_T w(t) f(t) \, dt = \sum_{i=1}^{n} \lambda_i f(t_i),$$

where $\lambda_i \geqslant 0$, and $t_i \in T$.

Example 3.3. Let $\Phi_1(z), \ldots, \Phi_n(z)$ be n complex, continuous, linearly independent functions, defined on a compact disk T in the complex plane. Assume $C_n(T)$ satisfies the generalized Krein condition. Then, there exist representations, exact for all $f \in C_n(T)$, of the form

$$\int_T f(z) \, dx \, dy = \sum_{i=1}^{2n} \lambda_i f(t_i), \qquad \lambda_i \geqslant 0, \; t_i \in T,$$

$$\int_C f(z) \, ds = \sum_{i=1}^{2n} \lambda_i f(t_i), \qquad \lambda_i \geqslant 0, \; t_i \in T,$$

where C is a rectifiable arc in T. However, in general,

$$\int_C f(z) \, dz$$

is not a nonnegative linear functional, and so, a corresponding representation does not exist.

The theory of nonnegative linear functionals is useful in numerical analysis, particularly from the viewpoint of existence of formulas with desired properties. It has also some quite practical applications, see Davis [*2*], or Wilson [*10*]. Further applications will be given in later papers.

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